

Laplace's method in Bayesian inverse problems

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Abstract

In a Bayesian inverse problem setting, the solution consists of a posterior measure obtained by combining prior belief, information about the forward operator, and noisy observational data. This measure is most often given in terms of a density with respect to a reference measure in a high-dimensional (or infinite-dimensional) Banach space. Although Monte Carlo sampling methods provide a way of querying the posterior, the necessity of evaluating the forward operator many times (which will often be a costly PDE solver) prohibits this in practice. For this reason, many practitioners choose a suitable Gaussian approximation of the posterior measure, in a procedure called Laplace's method. Once generated, this Gaussian measure is a lot easier to sample from and properties like moments are immediately acquired. This paper derives Laplace's approximation of the posterior measure attributed to the inverse problem explicitly as the posterior measure of a second-order approximation of the data-misfit functional, specifically in the infinite-dimensional setting. By use of a reverse Cauchy-Schwarz inequality we are able to explicitly bound the Hellinger distance between the posterior and its approximation.

1 Introduction

We consider an inverse problem

$$y = G(u) + \eta \quad (1)$$

where $G : X \rightarrow Y$ is a (possibly) nonlinear mapping between Hilbert spaces X, Y and $\eta \sim N(0, \Gamma)$ is additive noise. The challenge consists of inferring the value of u from the noisy (and maybe lower-dimensional) observation y . This is generally an ill-defined problem, so some sort of regularization is needed.

In a Bayesian approach, we assume that $u \sim \mu_0 = N(0, C_0)$, i.e. we have a Gaussian prior on the variable u . For simplicity we assume that the mean is 0, but this assumption can be dropped with slight modifications. This acts as a regularization and makes the inverse problem well-defined: Standard theory (see [15]) yields the posterior measure μ on u given an observation y under mild assumptions on the forward operator G :

$$\frac{d\mu}{d\mu_0}(u) = \frac{\exp(-\Phi(u))}{\int \exp(-\Phi(u)) d\mu_0(u)}, \quad (2)$$

where $\Phi(u) = \frac{\|y - G(u)\|_{C_0}^2}{2}$. Especially in higher dimensions, the posterior is often approximated by a suitable Gaussian, in order to make computation feasible. For a beautiful example how this is done in practice, see [1] in the setting of optimal experimental design of an infinite-dimensional inverse problem, or in a finite-dimensional context in [10]. This is called Laplace's method and is the focus of this work. This is done by defining the functional $I(u) = \Phi(u) + \frac{u^2}{2\sigma^2}$. The *maximum a posteriori point* is

$$u_{\text{MAP}} := \operatorname{arginf}_u I(u) \quad (3)$$

and the Laplace approximation is defined as

$$\nu = N(u_{\text{MAP}}, HI(u_{\text{MAP}})^{-1}) \quad (4)$$

This means that the Laplace approximation is the Gaussian measure centered at the Maximum A Posteriori point, with covariance operator matching the “local” covariance structure of the posterior measure. In finite dimensions, its density is exactly the normalized exponential of the measure's lognormal local quadratic approximation. For a good explanation (and as a general recommendation for a truly enjoyable book) of Laplace's method in finite dimensions, see [12]. A treatise about approximation of measures by Gaussian measures can be found in [13], although they employ the Kullback-Leibler divergence (or relative entropy) as a notion of distance between measures. [8] is an extensive study of various Gaussian approximation methods (including Laplace's method) in the context of reservoir modelling. Standard results about Laplace's method are recorded in [17], [3] Newer results on Laplace's method can be found in [9]. Gaussian approximations in a different context have been treated in [2], [14] in the case of diffusion processes and in [11] in the context of molecular dynamics. [16] presents an approach running “anti-parallelly” to Laplace's method: Instead of approximating the posterior measure directly, they approximate the forward operator or the negative log-likelihood. They, too, bound the Hellinger distance between the true posterior and the resulting approximation.

It can be shown that $\mu = \nu$ if G is linear and the prior measure was Gaussian in the first place. Heuristically, the approximation is bad when the posterior measure is multimodal or has different tail properties than a Gaussian.

We are interested in deriving concrete error bounds for the approximation quality $\mu \approx \nu$. The Hellinger distance between probability measures lends itself to this cause. Given two measures μ, ν which are absolutely continuous w.r.t. another measure μ_0 , the Hellinger distance (which is independent of the choice of μ_0) between μ and ν is

$$(d_H(\mu, \nu))^2 = \frac{1}{2} \cdot \int \left(\sqrt{\frac{d\mu}{d\mu_0}} - \sqrt{\frac{d\nu}{d\mu_0}} \right)^2 d\mu_0 = 1 - \int \sqrt{\frac{d\mu}{d\mu_0}} \sqrt{\frac{d\nu}{d\mu_0}} d\mu_0 \quad (5)$$

The main conclusion of this paper is recorded informally in the following claims and stated and proven later.

Claim 1. *While the posterior measure is given via its density w.r.t. the prior by*

$$\frac{d\mu}{d\mu_0}(u) = \frac{\exp(-\Phi(u))}{\int \exp(-\Phi(u)) d\mu_0(u)},$$

Laplace's method yields a Gaussian approximation of the form $\nu = N(u_{\text{MAP}}, HI(u_{\text{MAP}})^{-1})$ and its density w.r.t. the prior is

$$\frac{d\nu}{d\mu_0}(u) = \frac{\exp(-T\Phi(u))}{\int \exp(-T\Phi(u)) d\mu_0(u)}. \quad (6)$$

Here, $T\Phi(u) = \Phi(u_{\text{MAP}}) + D\Phi(u_{\text{MAP}})(u - u_{\text{MAP}}) + \frac{1}{2}H\Phi(u_{\text{MAP}})[u - u_{\text{MAP}}, u - u_{\text{MAP}}]$, the second order Taylor approximation of the data-misfit functional Φ .

Claim 2. *If there is $K \in (0, 1)$ such that*

$$\left\| \exp\left(-\frac{\Phi(u)}{2}\right) - \exp\left(-\frac{T\Phi(u)}{2}\right) \right\|_{L^2(X, \mu_0)} \leq K \cdot \frac{\exp(-I(u_{\text{MAP}}))}{\sqrt[4]{\det(C_0^{1/2} \cdot HI(u_{\text{MAP}}) \cdot C_0^{1/2})}},$$

then the Hellinger distance between the posterior and its Laplace's approximation can be bounded:

$$d_H(\mu, \nu) \leq \frac{K}{\sqrt{1 + (1 - K)^2}}$$

The paper is organized as follows: First the more intuitive one-dimensional case is presented and representation (6) is derived. The approach taken can not be generalized to the infinite dimensional case, as we use densities with respect to a Lebesgue measure. Then the infinite-dimensional equivalent is proven, where we show the equality (6) directly by means of characteristic functions. The following section shows that the problem of bounding the Hellinger distance can be reduced to a reverse Cauchy-Schwarz inequality. After recording a few elementary results about reverse CS inequalities, we immediately obtain lemma 2 and also a more practical (but less tight) version of it.

2 The Laplace approximation in one dimension

We recall

$$\frac{d\mu}{d\mu_0}(u) = \frac{\exp(-\Phi(u))}{\int \exp(-\Phi(u)) d\mu_0(u)},$$

and furthermore for the Lebesgue measure λ in one dimension, its μ 's Laplace approximation, which is a Gaussian centered at the maximum a posteriori point, with variance equal to the inverse of the second derivative of I at this point:

$$\frac{d\nu}{d\lambda} = \sqrt{\frac{I''(u_{\text{MAP}})}{2\pi}} \exp\left(-\frac{I''(u_{\text{MAP}})}{2} \cdot (u - u_{\text{MAP}})^2\right), \quad (7)$$

hence

$$\frac{d\nu}{d\mu_0} = \frac{\frac{d\nu}{d\lambda}}{\frac{d\mu_0}{d\lambda}} = \sqrt{I''(u_{\text{MAP}})} \cdot \sigma \cdot \exp\left(-\frac{I''(u_{\text{MAP}})}{2} \cdot (u - u_{\text{MAP}})^2 + \frac{u^2}{2\sigma^2}\right). \quad (8)$$

As we mentioned, second-order Taylor approximations will play a leading role, hence we define

$$R(u) = I(u) - I(u_{\text{MAP}}) - \frac{I''(u_{\text{MAP}})}{2}(u - u_{\text{MAP}})^2, \quad (9)$$

the error term of the second order Taylor approximation in u_{MAP} . Note that the first-order term vanishes because of u_{MAP} being a minimum of I , which we assume to be C^2 . Interestingly, $R(u)$ is

also an error term for the second order approximation of Φ , albeit in a slightly different way:

$$\begin{aligned}
R(u) &= \Phi(u) + \frac{u^2}{2\sigma^2} - \Phi(u_{\text{MAP}}) - \frac{u_{\text{MAP}}^2}{2\sigma^2} - \frac{\Phi''(u_{\text{MAP}})}{2}(u - u_{\text{MAP}})^2 - \frac{1}{\sigma^2}(u - u_{\text{MAP}})^2 \\
&= \frac{u_{\text{MAP}} \cdot (u - u_{\text{MAP}})}{\sigma^2} + \Phi'(u_{\text{MAP}})(u - u_{\text{MAP}}) \\
&\quad + \left[\Phi(u) - \Phi(u_{\text{MAP}}) - \Phi'(u_{\text{MAP}})(u - u_{\text{MAP}}) - \frac{\Phi''(u_{\text{MAP}})}{2}(u - u_{\text{MAP}})^2 \right] \\
&= \Phi(u) - T_{u_{\text{MAP}}}^{(2)} \Phi(u).
\end{aligned}$$

Note that the terms in the second line vanish because of $0 = I'(u_{\text{MAP}}) = \Phi'(u_{\text{MAP}}) + \frac{u_{\text{MAP}}}{\sigma^2}$ and the second term is exactly the error between $\Phi(u)$ and its second order Taylor polynomial developed in u_{MAP} . It holds that $I'(u_{\text{MAP}}) = 0$ but in general $\Phi'(u_{\text{MAP}}) \neq 0$. With this definition of R , and especially

$$-\frac{I''(u_{\text{MAP}})}{2} \cdot (u - u_{\text{MAP}})^2 + \frac{u^2}{2\sigma^2} = -\Phi(u) + I(u_{\text{MAP}}) + R(u),$$

we obtain for the density of ν w.r.t. μ_0 :

$$\frac{d\nu}{d\mu_0} = \sqrt{I''(u_{\text{MAP}})} \cdot \sigma \cdot \exp(I(u_{\text{MAP}})) \cdot \exp(-\Phi(u) + R(u)).$$

An easy calculation shows

$$\begin{aligned}
\frac{\exp(-I(u_{\text{MAP}}))}{\sigma \sqrt{I''(u_{\text{MAP}})}} &= \int \exp\left(\frac{u^2}{2\sigma^2} - I(u_{\text{MAP}}) - \frac{I''(u_{\text{MAP}})}{2}(u - u_{\text{MAP}})^2\right) \mu_0(du) \\
&= \int \exp(-\Phi(u) + R(u)) \mu_0(du) = \int \exp(-T_{u_{\text{MAP}}}^{(2)} \Phi(u)) d\mu_0(u).
\end{aligned} \tag{10}$$

This also follows immediately from the normalization of $d\nu/d\mu_0$ and thus

$$\frac{d\nu}{d\mu_0}(u) = \frac{\exp(-\Phi(u) + R(u))}{\int \exp(-\Phi(u) + R(u)) \mu_0(du)} = \frac{\exp(-T\Phi(u))}{\int \exp(-T\Phi(u)) \mu_0(du)}. \tag{11}$$

3 The Laplace approximation in general Hilbert spaces

In this section, we will show claim 1 in the setting of a general (possibly infinite-dimensional) Hilbert space. In order to do this, we need a slight adaptation of a useful Gaussian integral calculation.

In Proposition 1.2.8 in [4], the authors prove the following: Let $\mu_0 = N(0, Q)$ be a Gaussian measure on a real Hilbert space H . Assume that M is a symmetric operator such that $\langle Q^{1/2} M Q^{1/2} u, u \rangle < \langle u, u \rangle$ for all $0 \neq u \in H$. Then for $b \in H$

$$\int_H \exp\left(\frac{1}{2} \langle Mu, u \rangle + \langle b, u \rangle\right) d\mu_0(u) = \frac{\exp\left(\frac{1}{2} |(1 - Q^{1/2} M Q^{1/2})^{-1/2} \cdot Q^{1/2} b|^2\right)}{\sqrt{\det(1 - Q^{1/2} M Q^{1/2})}}. \tag{12}$$

We will need a generalization of this formula, which follows from analytical continuation of the (real) Hilbert space's inner product to its complex extension. Recall that this continuation will not be positively definite anymore: $\langle \lambda_1 a + b, \lambda_2 c + d \rangle = \lambda_1 \lambda_2 \langle a, c \rangle + \lambda_2 \langle b, c \rangle + \lambda_1 \langle a, d \rangle + \langle b, d \rangle$. The following lemma is stated without proof as it follows immediately from the bilinearity of the analytical continuation stated above.

Lemma 1. Let $\mu_0 = N(0, Q)$ be a Gaussian measure on a real Hilbert space H . Assume that M is a symmetric operator such that $\langle Q^{1/2}MQ^{1/2}u, u \rangle < \langle u, u \rangle$ for all $0 \neq u \in H$. Then, with $L := Q^{1/2}(1 - Q^{1/2}MQ^{1/2})^{-1}Q^{1/2}$ and for $b_1, b_2 \in H$

$$\begin{aligned} \int_H \exp\left(\frac{1}{2}\langle Mu, u \rangle + \langle b_1 + ib_2, u \rangle\right) d\mu_0(u) \\ = \frac{\exp\left(\frac{1}{2}\langle Lb_1, b_2 \rangle + i\langle Lb_1, b_2 \rangle - \frac{1}{2}\langle Lb_2, b_2 \rangle\right)}{\sqrt{\det(1 - Q^{1/2}MQ^{1/2})}}. \end{aligned} \quad (13)$$

Now we can prove our main result:

Proposition 1. Consider the inverse problem 1 with prior μ_0 and posterior μ given by

$$\frac{d\mu}{d\mu_0} = \frac{\exp(-\Phi(u))}{\int \exp(-\Phi(u))d\mu_0}.$$

The functional $I(u) = \Phi(u) + \frac{1}{2}\|u\|_{C_0}^2$ is assumed to be C^2 in a neighborhood of $u_{\text{MAP}} = \arg\inf I(u)$. Then the Laplace approximation of μ given by

$$\nu = N(u_{\text{MAP}}, HI(u_{\text{MAP}})^{-1})$$

is equivalently defined by

$$\frac{d\nu}{d\mu_0} = \frac{\exp(-T\Phi(u))}{\int \exp(-T\Phi(u))d\mu_0},$$

where $T\Phi(u) = \Phi(u_{\text{MAP}}) + D\Phi(u_{\text{MAP}})(u - u_{\text{MAP}}) + \frac{1}{2}H\Phi(u_{\text{MAP}})[u - u_{\text{MAP}}, u - u_{\text{MAP}}]$ is the second order Taylor approximation of Φ generated in u_{MAP}

Proof. This is done by comparing the Fourier transform (or characteristic function) of both representations for ν . We calculate

$$\int \exp(-T\Phi(u))d\mu_0(u) = \frac{e^{-I(u_{\text{MAP}})}}{\sqrt{\det C^{1/2}I''(u_{\text{MAP}})C^{1/2}}} \quad (14)$$

and

$$\int \exp(i\langle \lambda, u \rangle - T\Phi(u))d\mu_0(u) = \frac{e^{-I(u_{\text{MAP}})} \cdot \exp\left(i\langle u_{\text{MAP}}, \lambda \rangle - \frac{1}{2} \cdot I''(u_{\text{MAP}})^{-1}[\lambda, \lambda]\right)}{\sqrt{\det C^{1/2}I''(u_{\text{MAP}})C^{1/2}}} \quad (15)$$

We show (15), as (14) follows from setting $\lambda = 0$. We use

$$-T\Phi(u) = R(u) - \Phi(u) = \frac{\|u\|_C^2}{2} - I(u_{\text{MAP}}) - \frac{1}{2}I''(u_{\text{MAP}})[u - u_{\text{MAP}}, u - u_{\text{MAP}}].$$

and write $J = HI(u_{\text{MAP}})$ and $v = u_{\text{MAP}}$ for brevity. Note that for a bilinear operator K we will

identify $K(w, z) = \langle Kw, z \rangle$. Then

$$\begin{aligned}
& \int \exp(i\langle \lambda, u \rangle - T\Phi(u)) d\mu_0(u) \\
&= \int \exp\left(i\langle \lambda, u \rangle + \frac{\|u\|_C^2}{2} - I(v) - \frac{1}{2}J[u - v, u - v]\right) d\mu_0(u) \\
&= e^{-I(v)} \cdot \int \exp\left(\langle i\lambda, u \rangle + \frac{1}{2}\langle C^{-1}u, u \rangle - \frac{1}{2}\langle J(u - v), u - v \rangle\right) d\mu_0(u) \\
&= e^{-I(v) - \frac{1}{2}\langle Jv, v \rangle} \cdot \int \exp\left(\langle Jv + i\lambda, u \rangle + \frac{1}{2}\langle (C^{-1} - J)u, u \rangle\right) d\mu_0(u).
\end{aligned}$$

This is formula (13) with $M = C^{-1} - J$ and $b_1 = Jv$, $b_2 = \lambda$. In this case, $1 - C^{1/2}MC^{1/2} = 1 - C^{1/2}(C^{-1} - J)C^{1/2} = C^{1/2}JC^{1/2}$ and thus $(1 - C^{1/2}MC^{1/2})^{-1/2} = J^{-1/2}C^{-1/2}$. Continuing,

$$\begin{aligned}
&= e^{-I(v) - \frac{1}{2}\langle Jv, v \rangle} \cdot \frac{\exp\left(\frac{1}{2}|J^{1/2}v|^2 + i\langle J^{1/2}v, J^{-1/2}\lambda \rangle - \frac{1}{2}|J^{-1/2}\lambda|^2\right)}{\sqrt{\det C^{1/2}JC^{1/2}}} \\
&= e^{-I(v)} \cdot \frac{\exp\left(i\langle v, \lambda \rangle - \frac{1}{2}\langle J^{-1}\lambda, \lambda \rangle\right)}{\sqrt{\det C^{1/2}JC^{1/2}}}
\end{aligned}$$

This proves equation (15) and it follows that the characteristic function of the measure $\tilde{\nu}$ defined by $\frac{d\tilde{\nu}}{d\mu_0} = 1/Z \cdot \exp(-T\Phi)$ fulfills

$$\begin{aligned}
\hat{\nu}(\lambda) &= \frac{\int \exp(i\langle \lambda, u \rangle) d\tilde{\nu}(u)}{Z} = \frac{\int \exp(i\langle \lambda, u \rangle) \cdot \exp(-T\Phi(u)) d\mu_0(u)}{\int \exp(i\langle \lambda, u \rangle) d\mu_0(u)} \\
&= \exp\left\{i\langle u_{\text{MAP}}, \lambda \rangle + \frac{1}{2}\langle HI(u_{\text{MAP}})^{-1}(u), u \rangle\right\},
\end{aligned}$$

i.e. $\tilde{\nu} = N(u_{\text{MAP}}, HI(u_{\text{MAP}})^{-1})$ as claimed. \square

As in the one-dimensional setting, we conclude

$$\frac{d\mu}{d\mu_0} = \frac{\exp(-\Phi(u))}{\int \exp(-\Phi(u)) d\mu_0(u)}$$

and

$$\frac{d\nu}{d\mu_0} = \frac{\exp(-T\Phi(u))}{\int \exp(-T\Phi(u)) d\mu_0(u)}.$$

With these expressions, we derive a bound on the Hellinger distance between μ and ν in the next section.

4 Hellinger distance

There are many notions of metrics and semi-metrics between measures, notably total variation, Hellinger, Wasserstein, Prokhorov and Kullback-Leibler. A survey of these and more probability metrics, including a detailed exposition of their relations is [7].

We choose the Hellinger distance, mainly because of its good analytic properties, its consistency with the total variation metric and because of the following lemma which allows us to bound the difference between expectations under the different measures in question:

Lemma 2 (part of Lemma 7.14 in [5]). *Let μ, μ' be two probability measures which are absolutely continuous w.r.t. another measure ν on a Banach space $(X, \|\cdot\|_X)$. Assume that $f : X \rightarrow E$, where $(E, \|\cdot\|)$ has second moments with respect to both μ and μ' . Then*

$$\|\mathbb{E}^\mu f - \mathbb{E}^{\mu'} f\| \leq 2\sqrt{\mathbb{E}^\mu \|f\|^2 + \mathbb{E}^{\mu'} \|f\|^2} \cdot d_H(\mu, \mu').$$

Recall that

$$d_H(\mu, \nu)^2 = 1 - \int \sqrt{\frac{d\mu}{d\mu_0}}(u) \sqrt{\frac{d\nu}{d\mu_0}}(u) d\mu_0(u).$$

Thus, with the results of the preceding sections,

$$\begin{aligned} d_H(\mu, \nu)^2 &= 1 - \frac{\int \exp\left(-\frac{\Phi(u) + T\Phi(u)}{2}\right) d\mu_0(u)}{\sqrt{\int \exp(-\Phi(u)) d\mu_0(u)} \sqrt{\int \exp(-T\Phi(u)) d\mu_0(u)}} \\ &= 1 - \frac{\langle \exp(-\frac{1}{2}\Phi), \exp(-\frac{1}{2}T\Phi) \rangle_{L^2(X, \mu_0)}}{\|\exp(-\frac{1}{2}\Phi)\|_{L^2(X, \mu_0)} \cdot \|\exp(-\frac{1}{2}T\Phi)\|_{L^2(X, \mu_0)}} \end{aligned} \quad (16)$$

Remark 1. From positivity of the exponential and the Cauchy-Schwarz-Bunyakowsky inequality it can easily be seen that $d_H(\mu, \nu) \in [0, 1]$.

We would like to bound the Hellinger distance, i.e. optimally we would like to prove something like $d_H(\mu, \nu) \leq \epsilon$ for some $\epsilon > 0$. This amounts to a reverse Cauchy-Schwarz inequality, or a statement of the kind

$$\langle e^{-\Phi/2}, e^{-(T\Phi)/2} \rangle_{L^2(X, \mu_0)} > (1 - \epsilon^2) \cdot \|e^{-\Phi/2}\|_{L^2(X, \mu_0)} \cdot \|e^{-(T\Phi)/2}\|_{L^2(X, \mu_0)}. \quad (17)$$

In the next section, we present a few elementary results about this kind of inequality.

5 A reverse Cauchy-Schwarz inequality

In this section, H will always be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The study of reverse Cauchy-Schwarz-Bunyakowsky inequalities is an active field of research by itself. We refer to [6] for further reading, although we will need only a very basic form of a reverse CSB inequality.

Lemma 3. *Let $f, g \in H$ and $D > 0$ with $\|f - g\|^2 \leq D \cdot (\|f\|^2 + \|g\|^2)$. Then*

$$\langle f, g \rangle \geq \frac{1 - D}{2} \cdot (\|f\|^2 + \|g\|^2) \geq (1 - D) \cdot \|f\| \cdot \|g\|. \quad (18)$$

Proof.

$$\begin{aligned}
\langle f, g \rangle &= \frac{1-D}{2} \cdot \|f\|^2 - \frac{1-D}{2} \cdot \|g\|^2 \\
&= \frac{1}{2} \|f\|^2 + \frac{1}{2} \|g\|^2 - \frac{1}{2} \|f-g\|^2 - \frac{1-D}{2} \cdot \|f\|^2 - \frac{1-D}{2} \cdot \|g\|^2 \\
&= \frac{1}{2} [D \cdot (\|f\|^2 + \|g\|^2) - \|f-g\|^2] \geq 0.
\end{aligned}$$

The last inequality in (18) is just Young's inequality in \mathbb{R} . \square

Remark 2. Lemma 3 can be thought of as a reverse Young's inequality (which gives for all $f, g \in H$ that $\langle f, g \rangle \leq \frac{1}{2} \|f\|^2 + \frac{1}{2} \|g\|^2$ and thus if the conditions on f, g as in lemma 3 are fulfilled, we can bound

$$\frac{1-D}{2} \cdot (\|f\|^2 + \|g\|^2) \leq \langle f, g \rangle \leq \frac{1}{2} \cdot (\|f\|^2 + \|g\|^2).$$

Lemma 4. Let $f, g \in H$ and $K \in (0, 1)$ with $\|f-g\| \leq K \cdot \|f\|$. Then

$$\langle f, g \rangle \geq \left[\frac{1-K}{1+(1-K)^2} \right] \cdot (\|f\|^2 + \|g\|^2) \geq 2 \cdot \left[\frac{1-K}{1+(1-K)^2} \right] \cdot \|f\| \cdot \|g\|.$$

Proof.

$$\begin{aligned}
\frac{\|f-g\|^2}{\|f\|^2 + \|g\|^2} &\leq \frac{\|f-g\|^2}{\|f\|^2 + (\|f\| - \|f-g\|)^2} = \frac{\|f-g\|^2}{2\|f\|^2 - 2\|f\|\|f-g\| + \|f-g\|^2} \\
&\leq \frac{\|f-g\|^2}{2\|f\|^2 - 2K\|f\|^2 + \|f-g\|^2} = 1 - \frac{(2-2K)\|f\|^2}{(2-2K)\|f\|^2 + \|f-g\|^2} \\
&\leq 1 - \frac{2(1-K)}{1+(1-K)^2}
\end{aligned}$$

And using lemma 3 (with $D = 1 - \frac{2(1-K)}{1+(1-K)^2}$), we obtain the result. \square

Remark 3. This is another form of a reverse Young's inequality, but with a different prerequisite: If $\|f-g\| \leq K \cdot \|f\|$, we have

$$\frac{1}{2} \cdot \left(1 - \frac{K^2}{1+(1-K)^2} \right) \cdot (\|f\|^2 + \|g\|^2) \leq \langle f, g \rangle \leq \frac{1}{2} \cdot (\|f\|^2 + \|g\|^2) \quad (19)$$

6 Conditions for good approximation

From lemma 4 and the expression for $d_H(\mu, \nu)$ in (16) (the Hellinger distance between the posterior measure μ and its Laplace approximation ν) we immediately obtain the following:

Proposition 2. With the assumptions of proposition 1, if for some $K \in (0, 1)$

$$\left\| \exp\left(-\frac{\Phi(u)}{2}\right) - \exp\left(-\frac{T\Phi(u)}{2}\right) \right\|_{L^2(X, \mu_0)} \leq K \cdot \frac{\exp(-I(u_{\text{MAP}}))}{\sqrt[4]{\det(C_0^{1/2} \cdot H I(u_{\text{MAP}}) \cdot C_0^{1/2})}}, \quad (20)$$

then

$$d_H(\mu, \nu) \leq \frac{K}{\sqrt{1 + (1 - K)^2}} \quad (21)$$

Proof. We use lemma 4, where $H = L^2(X, \mu_0)$, $f = \exp(-T\Phi/2)$ and $g = \exp(-\Phi/2)$. Note that $\|f\|_H^2 = \exp(-I(u_{\text{MAP}}))/\sqrt{\det(C_0^{1/2} \cdot HI(u_{\text{MAP}}) \cdot C_0^{1/2})}$ due to (14). Then we can set $1 - \varepsilon^2 = 2 \left[\frac{1-K}{1+(1-K)^2} \right]$, hence $\varepsilon = \frac{K}{\sqrt{1+(1-K)^2}}$ in equation (17). \square

The following corollary uses assumptions which can be checked more easily but is less strict.

Corollary 1. *Define*

$$K^2 := \frac{\sigma \sqrt{I''(u_{\text{MAP}})}}{\exp(-I(u_{\text{MAP}}))} \cdot \int \exp(-\min\{\Phi(u), T\Phi(u)\}) \cdot \min \left\{ \frac{|\Phi(u) - T\Phi(u)|^2}{4}, 1 \right\} \mu_0(du).$$

Then we have

$$d_H(\mu, \nu) \leq \frac{K}{\sqrt{1 + (1 - K)^2}}.$$

Proof. This is due to the elementary inequality $|e^{-x} - e^{-y}| \leq e^{-(x \wedge y)} \cdot (|x - y| \wedge 1)$ from which we obtain

$$\begin{aligned} \int |e^{-\Phi(u)/2} - e^{-T\Phi(u)/2}|^2 d\mu_0 &\leq \int e^{-(\Phi(u) \wedge T\Phi(u))} \cdot (|\Phi(u) - T\Phi(u)|^2/4 \wedge 1) d\mu_0 \\ &= K^2 \cdot \int e^{-T\Phi(u)} d\mu_0. \end{aligned}$$

\square

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